Bounds on the Librations of a Symmetrical Satellite

Ralph Pringle Jr *
Lockheed Missiles and Space Company, Palo Alto, Calif

This papeı deals with the motion of a symmetrical rigid body in a circular orbit about a centıal attracting body The rigid body is free to rotate about its symmetry axis The complete equilibrium solutions for the motion of the symmetry axis in a rotating cooıdinate frame are obtained along with the stability of the equilibriums and bounds on the motion of the symmetry axis The results are plotted in a space of the ineıtia parameters and the spin velocity; the bounds on the motion of the symmetry axis are given as plots of projections of the unit sphere on normal planes This problem is solved by analogy with Hill's solution of the restricted problem of thıee bodies and by using Poincaré's bifurcation theory The results given form a very useful pıelude to actual solutions for the motions as well as an example of the use of methods that aıe moıe geneıally applicable to mechanical pıoblems

I Introduction

THE problem of rigid body motions in a gravity field dates at least to Newton who explained the first-order precession of the earth on the basis of his new gravitational theory The precession and nutation of the earth later became important to astronomers in connection with the length of the sidereal day Hipparchus first observed the effect of these rigid librations of the earth before the birth of Christ It was, however, D'Alembert who first gave a complete formulation of the problem for the case of high spin (precession and nutation) Laplace, Poisson, and Tisserand all subsequently worked on the problem, and it comes to us largely in the form in which they left it These investigators had in mind the verification of Newtonian gravity theory by pure deduction; our interest is in the use of this theory to predict phenomena occurring in connection with artificial satellites We are, therefore, interested in the complete motions of the system rather than just the high-spin case (of interest in the case of the earth)

Since the first artificial earth satellites were contemplated, there has been an increasing interest in the attitude dynamics of rigid bodies in orbit Work on the case of a symmetrical rigid body in a circular orbit has been done by Thomson,[1] Beletskii,[2] and Auelmann [3] Thomson considered the stability of the small amplitude motion of the symmetry axis of a satellite relative to the normal to the orbit plane Beletskii treated the high-spin case of a satellite with gravity and aerodynamic perturbations Auelmann attacked the problem of the present paper, but established results only for the "zero-spin" case

In this paper, the author will endeavor tc show a complete picture of the equilibrium solutions, their stability, and regions of stability This type of analysis is, of course, preliminary to a complete solution including approximate librations and periodic solutions

The paper draws on the theory of bifurcations of Poincaré[4] and on Hill's use of the Jacobian integral in the lunar theory [5] The problem is interesting because it has 1) a physical motivation and use, 2) a simplicity inherent in two-degree-of-freedom cases, and 3) a wealth of interesting phenomena

II Equations of Motion

Consider a symmetrical rigid body and a rectangular coordinate system with three mutually perpendicular unit vectors (**1b, 2b, 3b**) oriented so that **3b** points along the axis of symmetry **1b** and **2b** are perpendicular and intersect **3b** orthogonally at the center of mass We use two coordinate systems shown in Figs 1–3 The **1, 2, ϒ** plane in these figures is assumed to define the orbit plane, and the angle nt denotes the true anomaly of the circular orbit In the θ, φ set of angles, θ and φ are successive counterclockwise rotations about the **3** and **−1b** axes, respectively, whereas in the θ_1, θ_2 set, θ_2 and θ_1 denote successive counterclockwise rotations about the **2** and **−1b** axes, respectively $\dot\psi$ denotes the angular velocity of a third (cyclic) rotation ψ about the **3b** axes; this is the "spin velocity" The body is specified by a moment of inertia C about the **3b** axis and moments of inertia A about the **1b** and **2b** axes The two separate sets of angles are employed because of the singularity of each set at a particular point of interest

We may now write the kinetic and potential energy of the body as

$$\begin{aligned} T &= (A/2)\{\dot\varphi^2 + (\dot\theta + n)^2 \cos^2\varphi\} + (C/2)\{\dot\psi + (\dot\theta + n)\sin\varphi\}^2 \\ &= (A/2)\{(\dot\theta_1 + n \sin\theta_2)^2 + (\dot\theta_2 \cos\theta_1 - n\cos\theta_2 \sin\theta_1)^2\} + \tfrac{1}{2}C\{\dot\psi + n\cos\theta_2\cos\theta_1 + \dot\theta_2 \sin\theta_1\}^2 \quad (1) \\ V &= \tfrac{3}{2}n^2(C - A)\cos^2\varphi\cos^2\theta \\ &= \tfrac{3}{2}n^2(C - A)\cos^2\theta_1 \sin^2\theta_2 \end{aligned}$$

It is convenient to define

$$l = p_\psi/C \qquad r = C/A \tag{2}$$

where $p_\psi = d/dt(\partial T/\partial\dot\psi) = 0$ p_ψ is, therefore, a constant equal to

$$p_\psi = C[\dot\psi + (\dot\theta + n)\sin\varphi] = C[\dot\psi + n\cos\theta_2\cos\theta_1 + \dot\theta_2 \sin\theta_1] \tag{3}$$

We may now use Lagrange's equations to find the motion of the satellite under the potential energy V These may be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i} \qquad i = 1,2,3 \tag{4}$$

where the q_i are generalized coordinates taken to be either φ, θ, ψ or θ_1, θ_2, ψ There exists a first integral for the preceding problem[6]; this is the Hamiltonian H (which is *not* the total energy in this problem):

$$H = (\partial T/\partial\dot\psi)\dot\psi + (\partial T/\partial\dot\varphi)\dot\varphi + (\partial T/\partial\dot\theta)\dot\theta - T + V \tag{5}$$

Using (4) we can easily verify that $dH/dt = 0$ and, therefore, that H is a constant of the motion In terms of the variable

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* Senior Scientist; formerly Research Assistant, Stanford University, Stanford, Calif Member AIAA

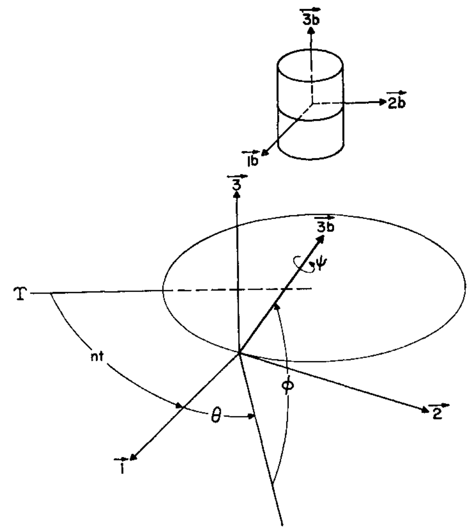

Fig 1 Coordinate system of a symmetrical rigid body in a circular orbit about the earth

of this problem, H becomes

$$H = R_2 + U \tag{6}$$

where

$$R_2 = (A/2)(\dot{\varphi}^2 + \dot{\theta}^2 \cos^2\varphi)$$

or

$$R_2 = (A/2)(\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 \cos^2\theta_1) \tag{7a}$$

$$U = \tfrac{3}{2}n^2A(r - 1)\cos^2\varphi \cos^2\theta - (An^2/2)\cos^2\varphi - Alrn \sin\varphi$$

or

$$U = \tfrac{3}{2}n^2A(r - 1)\cos^2\theta_1 \sin^2\theta_2 - Alrn \cos\theta_1 \cos\theta_2 - (An^2/2)(\cos^2\theta_2 \sin^2\theta_1 + \sin^2\theta_2) \tag{7b}$$

where the R_2 functions are positive definite functions of the velocities, and the U are functions of the coordinates φ, θ and θ_1, θ_2 Notice that $\dot{\psi}$ has been eliminated in favor of the constant l by use of (3) The function U is called the "dynamic potential" and plays an important role in the stability analysis that follows

III Equilibrium Points

An equilibrium point is a point in the φ, θ or θ_1, θ_2 space where the **3b** axis can come to rest relative to axes **1, 2, 3** This condition is defined, using (4), by

$$\partial U/\partial\varphi = U_\varphi = 0 \tag{8a}$$

$$\partial U/\partial\theta = U_\theta = 0 \tag{8b}$$

Using (7) and (8) gives the equilibrium relations

$$[3(1 - r)\sin\varphi_0 \cos^2\theta_0 + \sin\varphi_0 - (lr/n)]\cos\varphi_0 \tag{9a}$$

$$\cos\theta_0 \sin\theta_0 \cos^2\varphi_0 = 0 \tag{9b}$$

The preceding conditions reflect the static balance between gyroscopic and gravity torques We must investigate the equilibrium points φ_0, θ_0 defined by (9) We see that two cases obtain: I) $\varphi_0 = \pm\pi/2$, θ_0 arbitrary, and II) $\varphi_0 \neq \pm\pi/2$ Case II contains two subcases, i e , IIa) $\theta_0 = 0$, π, $\sin\varphi_0 = lr/n(4 - 3r)$, and IIb) $\theta_0 = \pm\pi/2$, $\sin\varphi_0 = lr/n$ We note that this, due to (9b), exhausts the possibilities

We must now discuss the stability of each point and whether U has a maximum, minimum, or saddlepoint Define the (hessian) matrices

$$\mathcal{H}_1 = \begin{bmatrix} U_{\theta_1\theta_1} & U_{\theta_1\theta_2} \\ U_{\theta_2\theta_1} & U_{\theta_2\theta_2} \end{bmatrix}$$

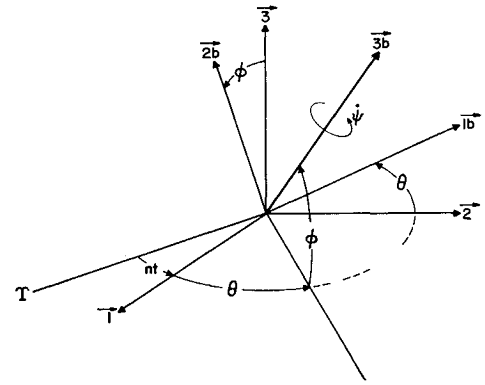

Fig 2 Rotations in ϕ, θ coordinates

$$\mathcal{H}_2 = \begin{bmatrix} U_{\varphi\varphi} & U_{\varphi\theta} \\ U_{\theta\varphi} & U_{\theta\theta} \end{bmatrix}$$

where the symbols $U_{q_iq_j} = \partial^2U/\partial q_i\partial q_j$ are evaluated at the various equilibrium points We shall use $\mathcal{H}_1$ for the points of case I and $\mathcal{H}_2$ for points of case II Since $\mathcal{H}$ can be thought of as the matrix of a quadratic form U in φ, θ, or θ_1, θ_2, when φ, θ, θ_1, θ_2 are small displacements from equilibrium, then: 1) if $\mathcal{H}$ is positive definite at φ_0, θ_0, or θ_{10}, θ_{20}, there exists a minimum of U, 2) if $\mathcal{H}$ is negative definite at the point of equilibrium, there exists a maximum of U, and 3) if $\mathcal{H}$ is sign variable at equilibrium, there exists a saddlepoint of U at the equilibrium point If $|\mathcal{H}| = 0$, there is a bifurcation at the equilibrium Note that it suffices to determine the results for $l > 0$ only This is so because in (7) U is unchanged if $l \to -l$ and $\varphi_0 \to -\varphi_0$

It can be shown[4 8] that a "point of bifurcation" occurs when the determinant of the hessian of U, $|\mathcal{H}|$, vanishes At a point of bifurcation, the qualitative nature of the stability behavior changes, e g , Figs 4–7 are the results of changing qualitative behavior This can be seen by using the fundamental result that $|\mathcal{H}| = \lambda_1\lambda_2$, where λ_1 and λ_2 are eigenvalues of the quadratic terms in the expansion of U about an equilibrium point Observe that, if $\lambda_1 > 0$, $\lambda_2 > 0$, we have a minimum of U; if $\lambda_1 > 0$, $\lambda_2 < 0$ or if $\lambda_1 < 0$, $\lambda_2 > 0$, we have a saddlepoint of U, and if $\lambda_1 < 0$, $\lambda_2 < 0$ we have a maximum of U This means that as $|\mathcal{H}| = \lambda_1\lambda_2$ passes through zero, the topology of U changes

Now consider the stability of points I and II by using the hessian matrix $\mathcal{H}$ to test U It can easily be seen that the elements of $\mathcal{H}_2$ are at θ_0, φ_0:

$$U_{\varphi\varphi} = +3n^2A(1 - r)\cos 2\varphi_0 \cos^2\theta_0 + An^2 \cos 2\varphi_0 + lrAn \sin\varphi$$

$$U_{\varphi\theta} = \tfrac{3}{4}n^2A(r - 1)\sin 2\varphi_0 \sin 2\theta_0$$

$$U_{\theta\theta} = \tfrac{3}{2}n^2A(r - 1)\cos^2\varphi_0 \cos 2\theta_0$$

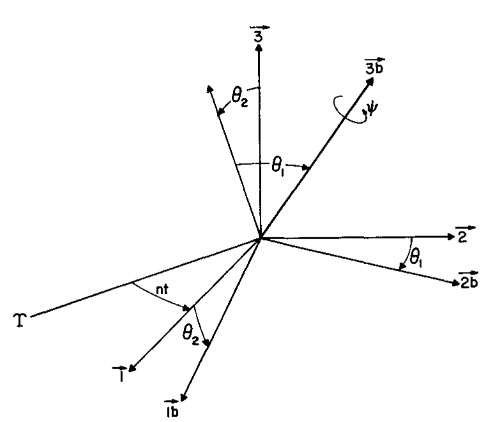

Fig 3 Rotations in θ_1, θ_2 coordinates

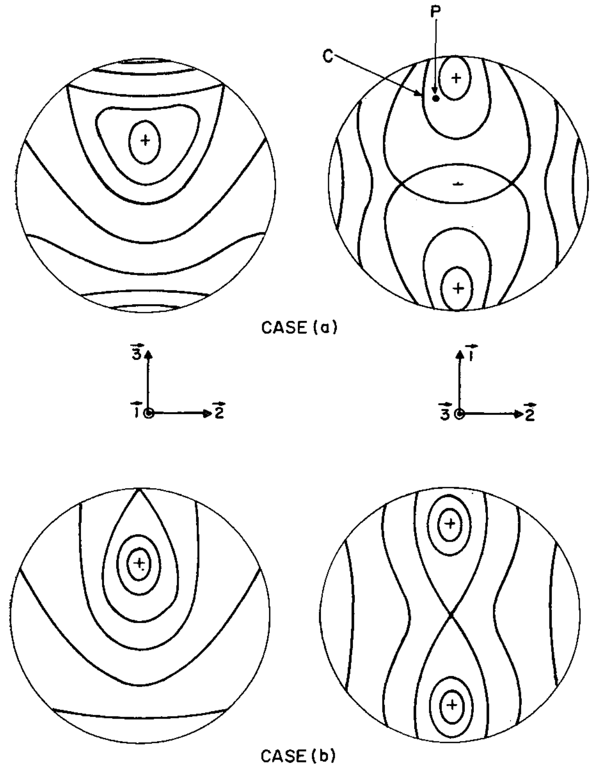

Fig 4 Curves of zero relative velocity

and at $\varphi_0 = \pi/2$, the elements of $\mathcal{H}_1$ become

$$U_{\theta_1\theta_1} = An^2[(lr/n) - 1]$$

$$U_{\theta_1\theta_2} = 0$$

$$U_{\theta_2\theta_2} = An^2[(lr/n) - 4 + 3r]$$

For case I it can be seen that, if $l/n > 1/r$, $l/n > (4/r) - 3$, the function U is minimum If only one of these is violated, U has a saddlepoint, and if $l/n < 1/r$, $l/n < (4/r) - 3$, then U has a maximum at $\varphi_0 = +\pi/2(\theta_{10} = 0,\ \theta_{20} = 0)$ For

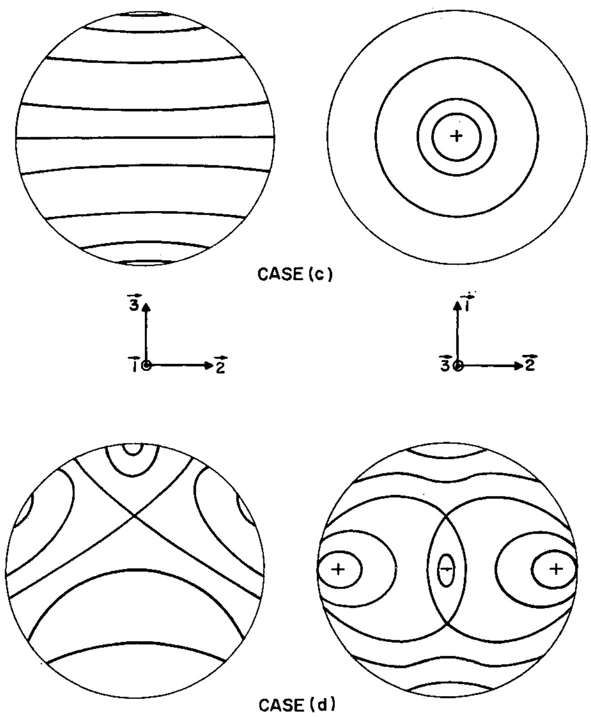

Fig 5 Curves of zero relative velocity

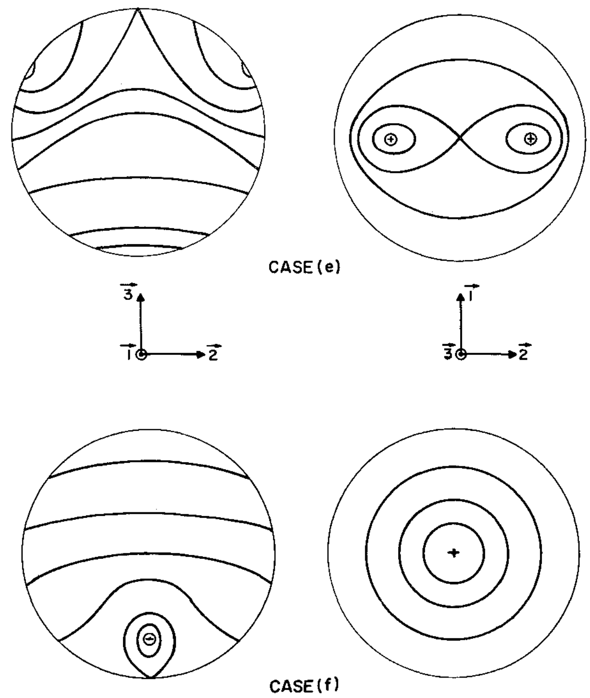

Fig 6 Curves of zero relative velocity

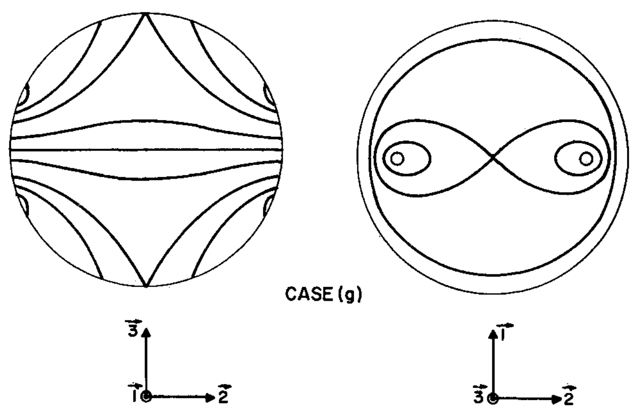

Fig 7 Curves of zero relative velocity

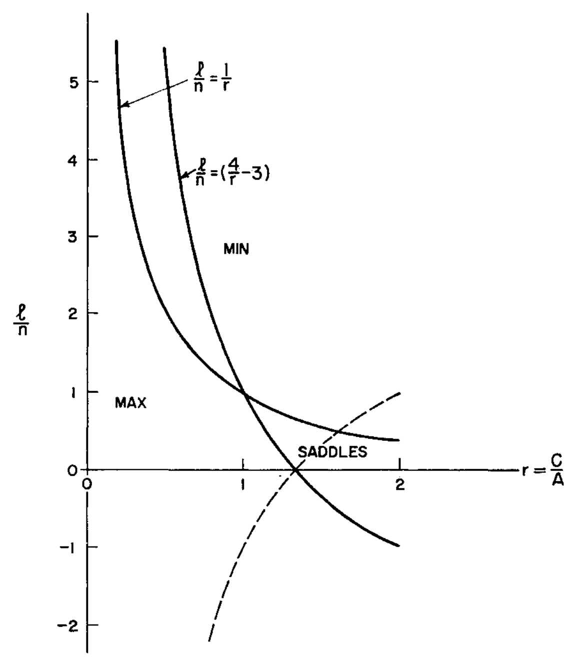

Fig 8 Case I: $\phi_0 = \pi/2$

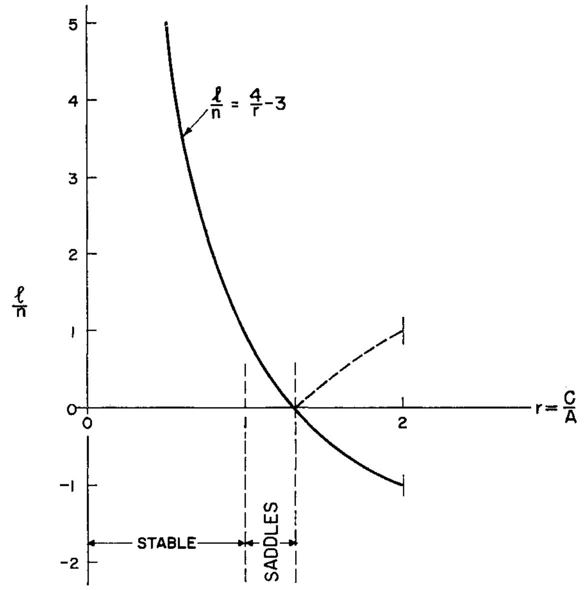

Fig 9 Case IIa: $\theta_0 = 0, \pi$; $\sin\phi_0$, $= (l/n)/[(4/r) - 3]$

case IIa [$\theta_0 = 0, \pi$; $\sin\varphi_0 = lr/n(4 - 3r)$] we see that, if $r < 1$, we have a stable (minimum) point If $1 < r < \frac{4}{3}$, we have a saddlepoint of U, and if $r > \frac{4}{3}$, we have a maximum of U There is a bifurcation at $r = 1$ For case IIb ($\theta_0 = \pm\pi/2$, $\sin\varphi_0 = lr/n$), $r = 1$ is a bifurcation point If $1 < r < 2$, there is a minimum of U, whereas if $0 < r < 1$, we have a saddlepoint of U These curves are plotted for the two cases in Figs 8–10 The solid curves in Fig 11 are "curves of bifurcation" This means they divide areas (of the plot of l/n vs r) with qualitatively different stability behavior In Figs 9 and 10, the solid lines are the curves beneath which the equilibrium point can exist These are derived by noticing that $\sin\varphi_0 \leq 1$ for existence of either equilibrium (IIa or IIb) Figure 11 is a composite plot showing all the curves of bifurcation on one plot For the purpose of sketching the behavior in configuration space, it is useful to know how many different kinds of qualitative behavior exist The curves of bifurcation divide the l/n, r plane in sectors (labeled *a–g*) If one moves across a curve of bifurcation, then he must use a new letter; thus, there are seven kinds of behavior in the quadrant $l > 0 (0 < \varphi_0 < \pi/2)$ and, likewise, seven for $l < 0$ $(-\pi/2 < \varphi_0 < 0)$

IV Curves of Constant *U*

It is now possible to plot contours of constant U in a space of φ, θ For the purpose of drawing them, it is useful to project the points of unit vector **3b** on the plane of **1**, **2** This is accomplished by the mapping $x = \cos\varphi \cos\theta$, $y = \cos\varphi \sin\theta$, where x is in the **1** direction (radial) and y is in the **2** direction (tangential)

It is useful to have a relation for when the slope $d\varphi/d\theta = 0$; this is, of course, the case of a contour parallel to the equator for which

$$d\varphi/d\theta = -(\partial U/\partial\theta)/(\partial U/\partial\varphi) = 0$$

if $U\varphi \neq 0$ From Eqs (8b) and (9b), $U_\theta = 0$ when $\theta = 0$, $\pm\pi/2$, π and when $\varphi = \pm\pi/2$ The point $\varphi = \pm\pi/2$ is never a point of zero slope because $\varphi = \pm\pi/2$ are always equilibrium points We then have $d\varphi/d\theta = 0$ when $\theta = 0, \pm\pi/2, \pi$, but $\theta = 0, \pm\pi/2, \pi$ are not equilibrium points This is useful in plotting curves of U = const

The curves of constant U are shown in Figs 4–7 for cases a–g of Fig 11 The points (+) are (stable) minima and those of (−) are maxima of U If initially the tip of the symmetry axis (**3b**) is within a closed contour, $U = H_0$; then, because $R_2 > 0$, we have $U \leq H_0$ for all subsequent

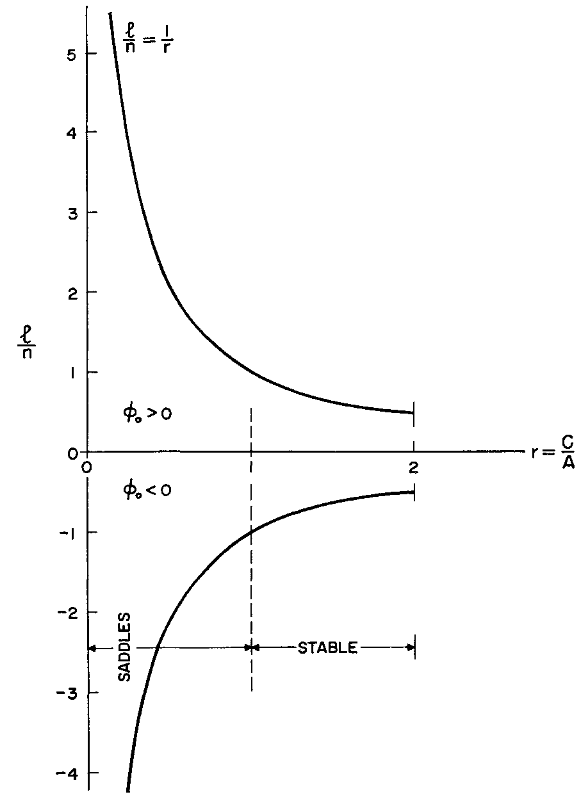

Fig 10 Case IIb: $\theta_0 = \pm\pi/2$; $\sin\phi_0 = lr/n$

motion If a closed contour surrounds a minimum point, then a motion exceeding $U = H_0 (U > H_0)$ would require $R_2 < 0$, which is impossible This proves that if in Fig 4a we have an initial position P, and if the initial velocities give an $H = H_0$, then the symmetry axis (**3b**) will always remain within the curve $C(U = H_0)$

The maximum regions of bounded motions about a minimum of U are surrounded by "separatrix" curves which always pass through saddlepoints If φ, θ is the position of a saddlepoint, then separatrix curves are defined by

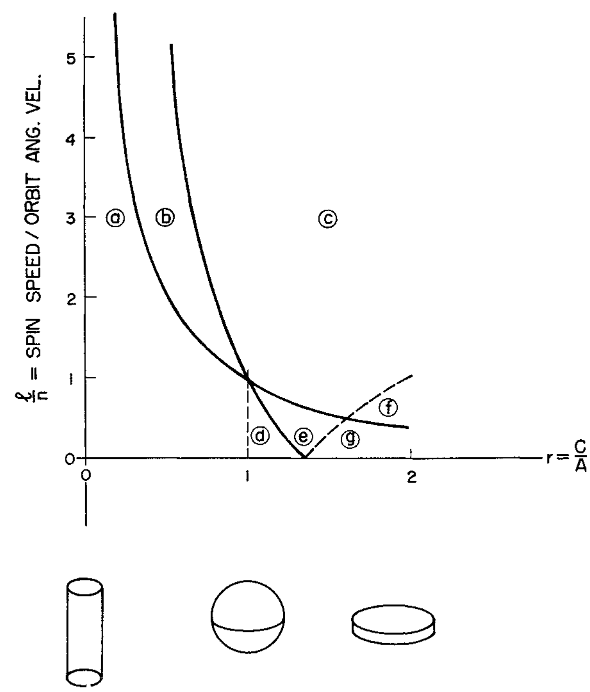

Fig 11 Curves of bifurcation

$$\cos^2\theta = \frac{3(r-1)\cos^2\varphi_s\cos^2\theta_s + \cos^2\varphi - \cos^2\varphi_s + \sin\alpha(\sin\varphi - \sin\varphi_s)}{3(r-1)\cos^2\varphi}$$

where $\sin\alpha = lr/n$, $\sin\beta = lr/n(4 - 3r)$ For the cases where separatrices exist, we have

Case a:

$$\theta_s = \pm\pi/2 \qquad \varphi_s = \alpha$$

Case b:

$$\varphi = \pi/2$$

Case d:

$$\theta = 0, \pi \qquad \varphi = \beta$$

Case e:

$$\varphi = \pi/2$$

Case f:

$$\varphi = -\pi/2$$

Case g:

$$\varphi_s = \pm\pi/2$$

The curves surrounding a maximum point of U tell us nothing about stability, but we can show, using small oscillation theory, that such a point may be stabilized if l/n is large enough For instance, Thomson and Kane[1 7] have shown that there are points of region (a) that are stable The condition that must hold for this to occur in case (a) is, for small oscillations,

$$(lr/n)^2 - 2(lr/n) + (3r - 1) > 0 \tag{10}$$

An example is $l/n = 3$, $r = 0\ 1$, which is stable in the small region and also lies in the "maximum of U" region of φ, θ For large motions in region (a), the nonlinear librations about $\varphi_0 = \pi/2$ must be investigated This has not been done

It is well known, as reflected in (10), that gyroscopic forces can stabilize a system of two degrees of freedom with such a potential maximum as occurs in region (a) The curves surrounding a maximum point of U only provide bounds on the nearness of approach to equilibrium; this is not too helpful One might be apprehensive about the maintenance of stability around a maximum of U in the presence of damping; this apprehension is justified by recourse to Lyapunov's stability theory It can be shown that, if damping (energy dissipation) occurs due to forces which do not effect the mass distribution of the satellite (e g , control jets or magnetic losses due to the earth's field) or to forces of internal damping involving elements with low mass and inertia, then the stability of the symmetry axis will occur if and only if at the equilibrium point U has a relative minimum[3] This indicates that, for engineering purposes, stability only occurs in those situations where U has a relative minimum It must, however, be stressed that each satellite system should be analyzed completely on its own Results from the preceding analysis should be applied only to the system defined in II The general approach of this paper is applicable to a wide class of systems with and without damping; the extensions to these cases are given in Ref 8

V Discussion of the Results

The results of this paper are presented in Figs 4–7 and 11 Figure 11 is the parameter plane of spin velocity vs the moment of inertia ratio The curves represent "curves of bifurcation' ; if one of these curves is crossed, the qualitative nature of the stability of equilibrium points changes The seven regions separated by the curves of bifurcation are shown in Figs 4–7 Cases (a) and (b) are for long, slim bodies, and, in these cases, the symmetry axis tends to point along the radius vector to the earth It is displaced upward due to spin and, therefore, appears as in Fig 4 Case (c) is for a short, flat body; this case has an equilibrium solution normal to the orbit plane as in Fig 5 Cases (d) and (e) are low spin, short, flat bodies and tend to point with their symmetry axes tangent to the orbit Cases (f) and (g) possess the property of having equilibriums in the lower hemisphere of the unit sphere at $\varphi_0 \neq -\pi/2$; otherwise case (f) is similar to case (c) and case (g) is similar to case (e)

The technique used to find the stability regions can be used to arrive at the regions for more complicated systems, but the geometry is more difficult in several dimension spaces To get the maximum U boundary curve surrounding a particular minimum point, however, is fairly simple If all the equilibriums are located (not always simple to solve the transcendental equations), then their value of U can be found Starting at a minimum point, we simply seek the lowest U saddle curve with energy greater than that of the minimum point in question; this is the lowest energy separatrix curve for the equilibrium point in question

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